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Du Pont's Paradox and Intensional Logic

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Abstract: Du Pont's paradox shows the failure of intensional logic to capture intensionality. The problem is to know what is the *Sinn* of a statement. One solution can be found in connection with the new classificiation of mathematics raised by Bourbaki.

The paradox of Du Pont

The following principle is well-known:

Principle of replacement: if $A \simeq B$ then $C[A] \simeq C[B/A]$.

Now consider a logic where there is a unary operator $\Box$ and in which this principle holds, in particular we have:

if $A \simeq B$ then $\Box(A \simeq B) \simeq \Box(A \simeq A)$.

Example:

From: THE AXIOM OF CHOICE is equivalent to ZORN'S LEMMA

we have: Du Pont believes that THE AXIOM OF CHOICE is equivalent to ZORN'S LEMMA

is equivalent to

Du Pont believes that THE AXIOM OF CHOICE is equivalent to THE AXIOM OF CHOICE

or

Du Pont has proved that THE AXIOM OF CHOICE is equivalent to ZORN'S LEMMA

is equivalent to

Du Pont has proved that THE AXIOM OF CHOICE is equivalent TO THE AXIOM OF CHOICE

What can we say about this paradox?

The present validity of the principle of replacement seems incompatible with the analysis of operators like "to believe", "to think", etc. If logics which deal with such kind of operators are called *intensional logics*, the principle of replacement must not be valid in intensional logics. Thus it seems strange that in most of the so-called intensional logics studied in the literature the principle of replacement holds. In which way these logics really deserve the name? What is intensionality?

The Paradox of George IV

Du Pont's paradox is strongly connected with another paradox concerning another famous gentleman which has been described by B. Russell in 'On Denoting':

> George IV wished to know whether Scott was the autor of *Waverley*; and in fact Scott *was* the autor of *Waverley*. Hence we may substitute *Scott* for *the author of 'Waverley'*, and thereby prove that George IV wished to know whether Scott was Scott. Yet an interest in the law of identity can hardly be attributed to the first gentleman of Europe.

How can we analyse this paradox? Frege explains the difference between $a=a$, and $a=b$ saying that a and b have the same *Bedeutung* but a different *Sinn*. If someone wants to know if $4+3=7$, he wants to know whether $4+3$ and 7, which have a different *Sinn*, have the same *Bedeutung*.

If George IV wants to know if THE AUTHOR OF WAVERLEY is SCOTT he wants to know whether THE AUTHOR OF WAVERLEY and SCOTT, which have a different *Sinn*, have the same *Bedeutung*. We can say that "wants to know" is intensional is the sense that George IV considers the relation between THE AUTHOR OF WAVERLEY and SCOTT not only from the point of view of their *Bedeutung*.

The solution of the paradox according to Frege is that we must not replace, in such kind of context, THE AUTHOR OF WAVERLEY or SCOTT by objects which have the same *Bedeutung* but only by objects which have the same *Sinn*. Now the difference between the paradox of George IV and the paradox of Du Pont is that in one we are concérned with proper names and in the other with statements. The problem is to know if we can treat statements and proper names in a strictly parallel way, to know what is the Sinn and the *Bedeutung* of a statement.

Sinn und Bedeutung in mathematical logic

The answer of Frege is that the *Bedeutung* of a statement (*Satz*) is a truth-value (*Wahreitswerht*) and its *Sinn* is a thought (*Gedanke*). The development of modern logic has been influenced by this distinction but in fact the solution is quite different.

What are the *Sinn* and *Bedeutung* of a statement in mathematical logic?

We can say that the *Sinn* is the position of the statement in the morphological algebra (in the case of propositional logic: the absolute free algebra of propositions) and that the *Bedeutung* is the set of models of the statement (in the case of propositional logic: the set of truth-functions which give the value 1 to the statement).

Extensional Wittgenstein

Let us consider the following item of the *Tractacus:* "If p follows from q and q from p, they are one and the same statement" (5.141). "p follows from q and q from p" can be interpreted as "p and q have the same *Bedeutung*" according to our definition of *Bedeutung* and *Tractacus* 5.101, 5.11, 5.12, 5.121 where Wittgenstein defines "follows". We call the *principle of extensionality*, the following interpretation of 5.141:

$$\text{if mod } A = \text{mod } B \quad \text{then } A = B$$

This principle means that we do not distinguish statements which have the same *Bedeutung*. Wittgenstenin's extensionality is radical because the notion of *Sinn* is totally rejected. This is done by the *principle of designation*:

> The identity of the object, I will express it by the identity of the sign and not by means of the identity sign. The difference between objects by the difference between signs. (5.530)

According to this principle 4+3 is not identical to 7, or THE AUTHOR OF WAVERLEY to SCOTT. From this point of view, 5.141 means that Wittgenstein considers the quotient algebra (Lindenbaum-Tarski algebra). For doing this the relation of identity must be compatible. Can we think that an *intensional logic* is a logic in which extensional identity is not compatible, i.e. in which we cannot identify statements which have the same *Bedeutung*? This may appear as a necessary condition but not at all as a sufficient condition.

Bourbaki's new classification of mathematics

The problem is that the notion of *Sinn* defined in a morphological way is too weak. The difference between THE AXIOM OF CHOICE and ZORN'S LEMMA does not lie in their syntactical layout but rather in the fact that they correspond to two different ways of thinking, one involving the notion of function and the other the notion of order. The problem is how these "ways of thinking" can be expressed.

One solution can be found in the idea of Bourbaki concerning fundamental structures. The new classification of mathematics raised by Bourbaki is based on the distinction between three kinds of structures: structures of order, topological structures, algebraic structures. These three different kinds of structures are like three different colours from which all mathematics can be painted. However the essence of this distinctions is not clear. The book of Bourbaki was in fact written in a spirit of an extremist syntacticalism in order to provide a non circular foundation of mathematics.

But no doubt that this idea of Bourbaki is independant of this philosophical mood and one interesting point would be to resettle it as to use it for a better understanding of the notion of *Sinn* and to develop a real intensional logic. [1]

Bibliography

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[1]This paper was written when the author was at the University of Wroclaw (Department of Logic) supported by a grant of the Polish Ministry of Education in the setting of France-Poland cooperation directed in France by the Ministry of Foreign Affairs.